Noncommutative quantum field theory: a confrontation of symmetries

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ABSTRACT: The concept of a noncommutative field is formulated based on the interplay between twisted Poincaré symmetry and residual symmetry of the Lorentz group. Various general dynamical results supporting this construction, such as the light-wedge causality condition and the integrability condition for Tomonaga-Schwinger equation, are presented. Based on this analysis, the claim of the identity between commutative QFT and noncommutative QFT with twisted Poincaré symmetry is refuted.

Contents

1. Introduction

Symmetry principles are invaluable guiding tools in the formulation of physical theories. Lie-group based symmetries proved their full worth in the construction of relativistic quantum field theory, in gauge field theories - actually in all the experimentally proven theories that are known. The more exotic quantum groups have been exhaustively studied starting with the 80ties; they have been mostly explored in various deformations of quantum mechanical systems, but not in the formulation of field theories. It is therefore natural that one particular deformation used in quantum groups, the twist, has become very popular since the twisted Poincaré algebra has been put in connection with the actively studied noncommutative field theories [1]. The connection between noncommutative space-time [2, 3] and quantum symmetry has its precursors in the context of string theory and quasitriangular Hopf algebras [4], followed shortly by an approach in the dual language of Hopf algebras [5]. What especially attracted interest when the twisted Poincaré algebra was rediscovered as a symmetry of noncommutative space-time was the realization that its representation content is the same as the one of the usual Poincaré algebra [1]. At the time there was a well-known problem concerning the representation content of the theory, which was now solved by the discovery of this new symmetry. The problem, as perceived earlier (see, for example, [6]), was that NC QFT on four-dimensional space-time was known to be symmetric under a subgroup of the Lorentz group, $SO(1,1) \times SO(2)$ (in case the time is noncommutative) or $O(1,1) \times SO(2)$ (which contains also reflection and is valid in case time is commutative) [7], which are both Abelian groups and thus have only one-dimensional

irreducible representations. Thus, the notion of spin seemed to be irremediably lost in NC QFT. Twisted Poincaré algebra rescued the spin of the representations and, moreover, indicated that one-particle irreducible representations retain the same classification in terms of mass and spin as in the case of Poincaré symmetry. Twisted Poincaré symmetry became thus a new concept of relativistic invariance for NC QFT [8].

Another thing that made the twist deformation very alluring was its simplicity. For the consistency of argumentation proposed in this paper we shall repeat a few main formulas of the construction of the twisted Poincaré algebra (for details on twist deformations and other quantum group techniques, see refs. [9–11]). The twisted Poincaré algebra is the universal enveloping of the Poincaré algebra $\mathcal{U}(\mathcal{P})$, viewed as a Hopf algebra, deformed with the Abelian twist element [12]

$$\mathcal{F} = \exp\left(\frac{i}{2}\theta^{\mu\nu}P_\mu \otimes P_\nu\right),\tag{1.1}$$

where $\theta_{\mu\nu}$ is a constant antisymmetric matrix (and not a tensor, i.e. it does not transform under the Lorentz transformations) and P_μ are the translation generators. This induces on the algebra of representations of the Poincaré algebra the deformed multiplication,

$$m \circ (\phi \otimes \psi) = \phi\psi \rightarrow m_\star \circ (\phi \otimes \psi) = m \circ \mathcal{F}^{-1}(\phi \otimes \psi) \equiv \phi \star \psi,\tag{1.2}$$

which is precisely the well-known Weyl-Moyal $\star$-product (taking the Minkowski space realization of P_μ, i.e. $P_\mu = -i\partial_\mu$):

$$\star = \exp\left(\frac{i}{2}\theta^{\mu\nu}\overleftarrow{\partial}_\mu \overrightarrow{\partial}_\nu\right).\tag{1.3}$$

The twist (1.1) does not affect the actual commutation relations of the generators of the Poincaré algebra $\mathcal{P}$:

$$\begin{aligned}
[P_\mu, P_\nu] &= 0, \\
[M_{\mu\nu}, P_\alpha] &= -i(\eta_{\mu\alpha}P_\nu - \eta_{\nu\alpha}P_\mu), \\
[M_{\mu\nu}, M_{\alpha\beta}] &= -i(\eta_{\mu\alpha}M_{\nu\beta} - \eta_{\mu\beta}M_{\nu\alpha} - \eta_{\nu\alpha}M_{\mu\beta} + \eta_{\nu\beta}M_{\mu\alpha}).
\end{aligned}\tag{1.4}$$

Consequently also the Casimir operators remain the same and the representations and classifications of particle states are identical to those of the untwisted Poincaré algebra.

However, the twist deforms the action of the generators in the tensor product of representations, or the so-called *coproduct*. In the case of the usual Poincaré algebra, the coproduct $\Delta_0 \in \mathcal{U}(\mathcal{P}) \times \mathcal{U}(\mathcal{P})$ is symmetric,

$$\Delta_0(Y) = Y \otimes 1 + 1 \otimes Y,\tag{1.5}$$

for all the generators $Y \in \mathcal{P}$. The twist $\mathcal{F}$ deforms the coproduct Δ_0 to $\Delta_t \in \mathcal{U}_t(\mathcal{P}) \times \mathcal{U}_t(\mathcal{P})$ as:

$$\Delta_0(Y) \longmapsto \Delta_t(Y) = \mathcal{F}\Delta_0(Y)\mathcal{F}^{-1}.\tag{1.6}$$

This similarity transformation is compatible with all the properties of $\mathcal{U}(\mathcal{P})$ as a Hopf algebra, since $\mathcal{F}$ satisfies the twist equation:

$$\mathcal{F}_{12}(\Delta_0 \otimes id)\mathcal{F} = \mathcal{F}_{23}(id \otimes \Delta_0)\mathcal{F}, \tag{1.7}$$

where $\mathcal{F}_{12} = \mathcal{F} \otimes 1$ and $\mathcal{F}_{23} = 1 \otimes \mathcal{F}$.

The twisted coproducts of the generators of Poincaré algebra turn out to be [1]:

$$\Delta_t(P_\mu) = \Delta_0(P_\mu) = P_\mu \otimes 1 + 1 \otimes P_\mu, \tag{1.8}$$

$$\Delta_t(M_{\mu\nu}) = M_{\mu\nu} \otimes 1 + 1 \otimes M_{\mu\nu} \tag{1.9}$$
$$- \frac{1}{2}\theta^{\alpha\beta}\left[(\eta_{\alpha\mu}P_\nu - \eta_{\alpha\nu}P_\mu) \otimes P_\beta + P_\alpha \otimes (\eta_{\beta\mu}P_\nu - \eta_{\beta\nu}P_\mu)\right].$$

Thus the twisted coproduct of the momentum generators is identical to the primitive coproduct, eq. (1.8), meaning that translational invariance is preserved, while the twisted coproduct of the Lorentz algebra generators, eq. (1.9), is nontrivial, implying the violation of Lorentz symmetry.

Taking in (1.2) $\phi(x) = x_\mu$ and $\psi(x) = x_\nu$, one obtains:

$$[x_\mu, x_\nu]_\star = i\theta_{\mu\nu}. \tag{1.10}$$

This is the usual commutation relation of the Weyl symbols of the noncommuting coordinate operators $\hat{x}$,

$$[\hat{x}_\mu, \hat{x}_\nu] = i\theta_{\mu\nu}, \tag{1.11}$$

which is obtained in the Weyl-Moyal correspondence. Thus, the construction of a NC quantum field theory through the Weyl-Moyal correspondence is equivalent to the procedure of redefining the multiplication of functions, so that it is consistent with the twisted coproduct of the Poincaré generators (1.6) [1].

As such, one would expect that all the features obtained in the past for NC QFT, like the connection between topology of space-time and the UV behaviour [13], UV/IR mixing [14], the light-wedge causality condition [7, 15, 16], the preservation of CPT symmetry and spin-statistics relation [15], the formulation of noncommutative gauge theories with symmetry under $\star$-gauge transformations [17] obeying very strict rules [18] (see also [19, 20]), the Lorentz-symmetry violation of the S-matrix in interacting NC theory [21] etc. would be confirmed by the symmetry of NC QFT under twisted Poincaré algebra.

The alluring simplicity of the twist turned it into the key-concept based on which noncommutative (quantum) field theories and noncommutative gravity have lately been studied. The recipe for extending the twist to other symmetries, like gauge symmetries and diffeomorphism transformations, seemed also at hand: one had to consider a commutative model with a certain symmetry, extend that symmetry by the Poincaré algebra through a direct or semidirect product, and use the twist element (1.1) to deform the new enveloping algebra. The $\star$-product (1.3) would automatically appear instead of the usual multiplication, due to (1.2), and the result would be a noncommutative gauge theory, for instance, with twisted gauge symmetry.

Using the twist deformation by the Abelian twist (1.1) and also the prescription given above, new results appeared in the literature, contradicting all the above mentioned features of noncommutative quantum field and gauge theories: the UV/IR mixing allegedly disappeared [22]; the spin-statistics relation was claimed not to hold [23]; twisted diffeomorphisms seemed to provide the general coordinate transformations in noncommutative gravity constructed with an immutable coordinate-dependent $\star$-product [24]; noncommutative gauge theories appeared to be easily constructed with symmetry under any gauge group (not only $U_\star(n)$) and possessing any representations [25, 26]; until finally NC QFT seemed to have the usual, light-cone causality condition as well as Lorentz symmetry, and ultimately to be identical to commutative QFT [27–29].

The various controversies that ensued were resolved in favour of the traditional dynamical approach to NC QFT: the UV/IR mixing was shown still to be present [30, 31]; the spin-statistics relation was proven to hold [30, 32, 33]; the twisted gauge theories and implicitly the twisted diffeomorphisms were shown to be constructed in a manner inconsistent with the concept of gauge invariance [34, 35], thus leaving only the option of $\star$-gauge symmetry with its restrictions. The consistent use of the twist deformation technique turned out to support the dynamical calculations. We shall not return to these issues in this paper.

In this paper we shall show the intrinsic impossibility of the identity between noncommutative and commutative (quantum) field theory. We shall approach the subject from different points of view: a general argument, based on Pauli's Theorem; a general derivation of the causality condition in noncommutative interacting theories as integrability condition for the Tomonaga-Schwinger equation; and finally a new interpretation of the noncommutative field operator itself in a theory with twisted Poincaré symmetry. All these approaches will lead to the same conclusion, that the twisted Poincaré symmetry of noncommutative (quantum) field theory is reduced to the residual $O(1,1) \times \mathrm{SO}(2)$ symmetry, but still carrying representations of the full Lorentz group. Consequently, Lorentz invariance is absent and noncommutative QFT is in essence different from commutative QFT.

2. Lorentz invariance and Pauli's Theorem

In 1957, after learning that weak interactions violate parity, Pauli introduced what we shall call the Pauli group (not to be confused with the group of the σ-matrices!) in order to explain why the violation of parity had not been earlier recognized in beta-decay [36]. In our case we shall use not the Pauli group itself, but the philosophy behind it, as described in [37].

Let $\mathcal{L}(g_i; \psi_j)$ be the Lagrangian density of a system of fields, where ψ_j denotes the field operators and g_i a fundamental parameter, such as a mass, or a coupling constant, or - in our case - the noncommutativity parameter. Assuming that the field operators transform under a group G as

$$\psi_j \to \psi'_j, \tag{2.1}$$

the change in $\mathcal{L}$ caused by (2.1) can be compensated by a change in the parameters,

$$g_i \rightarrow g_i', \tag{2.2}$$

such that the Lagrangian density would be invariant,

$$\mathcal{L}(g_i; \psi_j) = \mathcal{L}(g_i'; \psi_j'). \tag{2.3}$$

An observable quantity will depend on the set of parameters in the Lagrangian, but not on the field operators. If G is a symmetry group of the system described by the Lagrangian $\mathcal{L}(g_i; \psi_j)$, then an observable $\mathcal{O}(g_i)$ must satisfy the condition:

$$\mathcal{O}(g_i) = \mathcal{O}(g_i'), \tag{2.4}$$

in other words, $\mathcal{O}(g_i)$ must be a function of g_i invariant under G.

The statement (2.4) will be called Pauli's Theorem in what follows. There is a question whether this theorem is valid for the Lorentz group or not. Practical calculations show that the S-matrix elements depend not only on Lorentz invariant combinations such as $\theta^{\mu\nu}\theta_{\mu\nu}$, but also on non-invariant $p^\mu\theta_{\mu\nu}k^\nu$, $p^\mu\theta_{\mu\alpha}\theta^{\alpha\nu}k_\nu$ etc., indicating violation of Pauli's Theorem. It is plausible, therefore, that Pauli's Theorem is valid only for an internal symmetry group G and for a finite set of parameters $\{g_i\}$, just like the Coleman-Mandula Theorem. The momenta p, k,... are not present in the original Lagrangian and they can not be included in the finite set of parameters, showing explicitly violation of Pauli's Theorem for the Lorentz group.[1]

Based on this general argument we have to conclude that Lorentz invariance is violated in NC QFT with twisted Poincaré symmetry, if the parameter $\theta_{\mu\nu}$ appears in the observables. The complete disappearance of $\theta_{\mu\nu}$ from the observables or its presence contracted only to itself would be the effect of a peculiar conspiracy of accidents.

In actual calculations performed in NC QFT the appearance of $\theta_{\mu\nu}$ contracted with momenta of particles is commonplace, and it is the reason for the emergence of UV/IR mixing [14] and of the light-wedge causality condition [7, 15, 16], to name only two essential aspects with far-reaching consequences, among which the failure of analyticity of the scattering amplitude [38] and the nonexistence of high-energy bounds of the Froissart-Martin type on the total cross-section in NC QFT [39] are representative examples.

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[1]If one erroneously applies this theorem to the Lorentz group one may come to the conclusion of Lorentz symmetry for NC QFT. For example, in ref. [27], such a conclusion was drawn in the axiomatic approach to NC QFT. While justly observing that the shifts of coordinates $\star$-commute among themselves and the noncommutative Wightman functions, as translationally invariant objects, depend only on shifts of coordinates, it was however overlooked that the shifts of coordinates contracted with $\theta_{\mu\nu}$ are also commuting variables which may (and will) appear in the Wightman functions. Indeed, by shifting the coordinates in a $\star$-product of functions, the θ-dependence does not vanish. Should the θ-dependence of the Wightman functions disappear by the shift of coordinates, it would mean that the requirement of translational invariance implies necessarily Lorentz invariance.

3. The light-wedge causality condition and the Tomonaga-Schwinger equation in NC QFT

In anticipation of the light-wedge causality condition, we shall consider that the constant matrix θ has no time-space components, i.e. $\theta_{0i} = 0$, compatible with causality [40] and unitarity [41]. Without loss of generality, we choose the coordinate system (which will be used throughout the paper) in such a way that the θ-matrix is written in the form:

$$\theta^{\mu\nu} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \theta \\ 0 & 0 & -\theta & 0 \end{pmatrix}, \tag{3.1}$$

i.e. $\theta_{23} = -\theta_{32} = \theta$, while all other components vanish. This configuration of the θ-matrix is invariant under the action of the subgroup $O(1,1) \times \mathrm{SO}(2)$ of the Lorentz group [7]. When appropriate, we shall comment on other possible configurations of the θ-matrix as well. We further use the notation

$$
\begin{aligned}
x^\mu &= (\tilde{x}, \mathbf{a}), \\
y^\mu &= (\tilde{y}, \mathbf{b}), \\
\tilde{x} &= (x^0, x^1),\ \tilde{y} = (y^0, y^1), \\
\mathbf{a} &= (x^2, x^3), \\
\mathbf{b} &= (y^2, y^3),
\end{aligned}
\tag{3.2}
\tag{3.3}
$$

and consider x^2, x^3 (and y^2, y^3) as internal degrees of freedom. We thus confine ourselves to one time and one space dimension.

In the following we shall use the integral representation for the Moyal $\star$-product, which reads, in general [42][2]

$$(f \star g)(x) = \int d^D y\ d^D z\ \mathcal{K}(x; y, z) f(y) g(z), \tag{3.4}$$

where

$$\mathcal{K}(x; y, z) = \frac{1}{\pi^D \det \theta} \exp[-2i(x\theta^{-1}y + y\theta^{-1}z + z\theta^{-1}x)], \tag{3.5}$$

with D being the even dimension of the invertible matrix θ, $\det \theta$ being its determinant and we use the notation $x\theta^{-1}y = x^\mu (\theta^{-1})_{\mu\nu} y^\nu$.

In our case, the invertible part of θ is a 2×2-submatrix in the $(2,3)$-plane and the integration goes only over the noncommutative coordinates, such that we can write the integral form of the $\star$-product of n functions as:

$$(f_1 \star f_2 \star \cdots \star f_n)(x) = \int d\mathbf{a}_1 d\mathbf{a}_2 \cdots d\mathbf{a}_n \mathcal{K}(\mathbf{a}; \mathbf{a}_1, \cdots, \mathbf{a}_n) f_1(\tilde{x}, \mathbf{a}_1) f_2(\tilde{x}, \mathbf{a}_2) \cdots f_n(\tilde{x}, \mathbf{a}_n), \tag{3.6}$$

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[2]Other integral representations exist as well, see, e.g., ref. [43], but the actual form of the integral representation is not important for our argument, as will be shortly seen.

where

$$\mathcal{K}(\mathbf{a}; \mathbf{a}_1, \cdots, \mathbf{a}_n) = \frac{1}{(\pi^2 \det \theta)^{n/2}} \exp[-2i(\mathbf{a}\theta^{-1}\mathbf{a}_1 + \mathbf{a}_1\theta^{-1}\mathbf{a}_2 + \cdots + \mathbf{a}_n\theta^{-1}\mathbf{a})]. \qquad (3.7)$$

The kernel (3.7) is SO(2) invariant.

Tomonaga-Schwinger equation in two dimensions. The Tomonaga-Schwinger equation [44, 45] (see also [46]) is the covariant generalization of the Schrödinger equation in the interaction picture, formulated as a functional differential equation incorporating arbitrary Cauchy surfaces, and not only those of constant Minkowski time.

In commutative QFT the Tomonaga-Schwinger equation reads:

$$i\frac{\delta}{\delta\sigma(x)}\Psi[\sigma] = \mathcal{H}_{\text{int}}(x)\Psi[\sigma], \qquad (3.8)$$

where $\mathcal{H}_{\text{int}}(x)$ is the interaction Hamiltonian density, and σ is a space-like surface (i.e. a surface whose every two points are space-like separated). The existence of solutions for the Tomonaga-Schwinger equation is insured if the integrability condition

$$\frac{\delta^2\Psi[\sigma]}{\delta\sigma(x)\delta\sigma(x')} - \frac{\delta^2\Psi[\sigma]}{\delta\sigma(x')\delta\sigma(x)} = 0, \qquad (3.9)$$

with x and x' on the surface σ, is satisfied. This integrability condition then implies

$$[\mathcal{H}_{\text{int}}(x), \mathcal{H}_{\text{int}}(x')] = 0. \qquad (3.10)$$

Since in the interaction picture the field operators satisfy free-field equations, they satisfy Lorentz invariant commutation rules. The Lorentz invariant commutation relations are such that (3.10) is satisfied automatically, since x and x' are space-like separated.

In the noncommutative case, the use of the interaction picture has the advantage that the free-field equations satisfied by the noncommutative fields are identical to the corresponding free-field equations of the commutative case. The Tomonaga-Schwinger equation in the noncommutative case will read:

$$i\frac{\delta}{\delta\mathcal{C}}\Psi[\mathcal{C}] = \mathcal{H}_{\text{int}}(x)_\star\Psi[\mathcal{C}], \qquad (3.11)$$

$$\mathcal{H}_{\text{int}}(x)_\star = \lambda[\phi(x)]^n_\star, \qquad (3.12)$$

where $\mathcal{C}$ is a 1-dimensional surface (i.e. a curve) embedded in the plane of commutative coordinates (x^0, x^1). The fields $\phi(x)$ satisfy free-field equations and the Hamiltonian of interaction is built up by $\star$-multiplying the fields.

The integrability condition is:

$$[\mathcal{H}_{\text{int}}(x)_\star, \mathcal{H}_{\text{int}}(y)_\star] = 0, \quad \text{for } x, y \in \mathcal{C}, \qquad (3.13)$$

which we can write as

$$\begin{aligned} &[(\phi \star \ldots \star \phi)(\tilde{x}, \mathbf{a}), (\phi \star \ldots \star \phi)(\tilde{y}, \mathbf{b})] \\ &= \int \prod_{i=1}^n d\mathbf{a}'_i \, \mathcal{K}(\mathbf{a}; \mathbf{a}'_1, \cdots, \mathbf{a}'_n) \int \prod_{i=1}^n d\mathbf{b}'_i \, \mathcal{K}(\mathbf{b}; \mathbf{b}'_1, \cdots, \mathbf{b}'_n) \\ &\qquad \times [\phi(\tilde{x}, \mathbf{a}'_1) \ldots \phi(\tilde{x}, \mathbf{a}'_n), \phi(\tilde{y}, \mathbf{b}'_1) \ldots \phi(\tilde{y}, \mathbf{b}'_n)] = 0. \qquad (3.14) \end{aligned}$$

The commutators of products of fields appearing in (3.14) are written as products of fields at various space-time points multiplied by invariant commutators of fields. A typical factor is

$$\phi(\tilde{x}, \mathbf{a}_1') \ldots \phi(\tilde{x}, \mathbf{a}_{n-1}') \phi(\tilde{y}, \mathbf{b}_1') \ldots \phi(\tilde{y}, \mathbf{b}_{n-1}') \big[\phi(\tilde{x}, \mathbf{a}_n'), \phi(\tilde{y}, \mathbf{b}_n')\big]. \tag{3.15}$$

The fields at every point are independent, since they are systems with an infinite number of degrees of freedom. As a result, their products will also be independent. Eq. (3.14) becomes a sum of independent products of fields, whose coefficients have to vanish identically in order for the whole sum to vanish. Since the kernel can not vanish, it remains as a necessary condition for the commutators of fields to be zero at every point,

$$\big[\phi(\tilde{x}, \mathbf{a}_i'), \phi(\tilde{y}, \mathbf{b}_j')\big] = 0. \tag{3.16}$$

This condition is satisfied outside of the mutual light-cone:

$$(x^0 - y^0)^2 - (x^1 - y^1)^2 - (a_i^{2'} - b_j^{2'}) - (a_i^{3'} - b_j^{3'})^2 < 0, \tag{3.17}$$

since all $\phi(x)$ satisfy the same free-field equations and the same invariant commutation relations as in the commutative case. However, $\mathbf{a}_i'$ and $\mathbf{b}_j'$ are integration variables in the range

$$0 \le (a_i^{2'} - b_j^{2'})^2 + (a_i^{3'} - b_j^{3'})^2 < \infty \tag{3.18}$$

and therefore the necessary condition becomes

$$(x^0 - y^0)^2 - (x^1 - y^1)^2 < 0, \tag{3.19}$$

i.e. the light-wedge causality condition, symmetric under the stability group of $\theta_{\mu\nu}$, $O(1,1) \times$ SO(2).

Remark that the light-wedge causality condition is obtained here in a general approach, without using the actual mode expansion of the fields, but only the fact that in the interaction picture the field operators satisfy free-field equations and the integral representation of the Moyal $\star$-product. In the appendix we show that the light-wedge configuration is obtained for any commutator of $\star$-products of field operators, starting from the commutator with the simplest powers of field operators.

We should point out that, were we to allow the time to be noncommutative, i.e. $\theta_{0i} \neq 0$ (in a Lorentz invariant manner), then time would have entered the $\star$-product and be integrated over in the integral representation (3.4). The time variable as an integration variable which can not be fixed would have crept in (3.17), resulting in the impossibility of deriving any causality condition. We can therefore conclude that quantum field theories with noncommutative time do not fulfil an integrability condition for the Tomonaga-Schwinger equation. Although some theories with noncommutative time may appear to have desirable properties, like unitarity or Lorentz symmetry, these constructions are jeopardized by the lack of solution of the Tomonaga-Schwinger equation, implying that the space of states in the interaction picture is empty.

The lack of causality is a problem also in certain theories in which $\theta_{\mu\nu}$ is a Lorentz tensor and the Moyal $\star$-product (1.3) is used [47]. The hope of having wedge-causality in a

theory with $\theta_{\mu\nu}$ transforming as a Lorentz tensor [48] may be deceptive: the shape of the wedge is given by the commuting coordinates, since the nonlocality in the noncommutative directions makes the speed of propagation of a signal infinite in those directions. For a wedge to exist it is essential that the time coordinate be commutative. Assuming that one starts with a system of reference in which $\theta_{\mu\nu}$ has a form similar to (3.1) and wedge-locality is apparent, since $\theta_{\mu\nu}$ is a genuine Lorentz tensor, one can always boost to a frame of reference in which $\theta_{\mu\nu}$ picks up time components. In the new system, time is noncommutative, consequently the wedge simply disappears. In order to transform a wedge into another by a Lorentz transformation, one has to discard all the transformations which give $\theta_{\mu\nu}$ nonvanishing time components, but this is to break the Lorentz symmetry from the beginning.

The light-wedge causality condition for NC QFT has been recently obtained in another general context, which is the axiomatic formulation. Without any reference to specific models, based only on the fact that the Wightman functions have to be defined in NC QFT with $\star$-products [49]:

$$W_\star(x_1, x_2, \ldots x_n) = \langle 0|\phi_1(x_1) \star \phi_2(x_2) \star \ldots \star \phi_n(x_n)|0\rangle \,,$$

it has been shown in [50] (see also [51]) that the space of test functions smearing these noncommutative Wightman functions is one of the Gel'fand-Shilov spaces S^β with $\beta < 1/2$. These test functions can have finite support only in the commutative directions (if such directions exist), therefore the local commutativity condition (or microcausality condition) which is central to the axiomatic approach can be formulated only with respect to the light-wedge.

4. Twisted Poincaré symmetry and the residual $O(1,1) \times \mathrm{SO}(2)$ invariance

The dynamical calculations performed or reviewed in this paper show that noncommutative quantum field theories with a constant noncommutativity parameter $\theta_{\mu\nu}$ break Lorentz invariance and, depending on the structure of the θ-matrix, retain a symmetry under the Lorentz subgroup $\mathrm{SO}(1,1) \times \mathrm{SO}(2)$ when $\theta_{0i} \neq 0$ or under $O(1,1) \times \mathrm{SO}(2)$ when $\theta_{0i} = 0$. The second case is of physical interest, since it avoids the known problems with causality [40, 7, 15] and unitarity [41], preserving the notion of light-wedge causality. General arguments, based on the philosophy of the Pauli group, also support these results. It is then natural to expect that a consistent construction based on the twisted Poincaré algebra leads to the same outcome[3]. A rigorous construction of noncommutative fields, starting from first principles and twisted Poincaré algebra, has only recently been put forward [53]. Obviously, the implications of twisted Poincaré symmetry on the content of one-particle irreducible representations should have bearing also on the definition of noncommutative fields, though this relation is not straightforward. Answering the question

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[3]"Symmetry is a tool that should be used to determine the underlying dynamics, which must in turn explain the success (or failure) of the symmetry arguments. Group theory is a useful technique, but it is no substitute for physics."(Howard Georgi, [52])

about what a noncommutative field is, in the sense of the actions of the twisted Poincaré algebra, will finally lead us to the explicit meaning of twisted Poincaré invariance in NC QFT.

4.1 $O(1,1) \times \mathrm{SO}(2)$ invariance from the perspective of the twist

Before moving further to the construction of a noncommutative fields, let us first consider simple facts relating the twisted Poincaré algebra $\mathcal{U}_t(\mathcal{P})$ and the residual symmetry $O(1,1) \times \mathrm{SO}(2)$. With the θ-matrix configuration (3.1), i.e. x^0 and x^1 as commutative coordinates and x^2 and x^3 as noncommutative coordinates, one can calculate the twisted coproducts of all the Lorentz generators according to the formula (1.9). The result is:

$$\Delta_t(M_{01}) = \Delta_0(M_{01}) = M_{01} \otimes 1 + 1 \otimes M_{01},$$
$$\Delta_t(M_{23}) = \Delta_0(M_{23}) = M_{23} \otimes 1 + 1 \otimes M_{23}\,, \tag{4.1}$$

while

$$\Delta_t(M_{02}) = \Delta_0(M_{02}) + \frac{\theta}{2}(P_0 \otimes P_3 - P_3 \otimes P_0),$$
$$\Delta_t(M_{03}) = \Delta_0(M_{03}) - \frac{\theta}{2}(P_0 \otimes P_2 - P_2 \otimes P_0),$$
$$\Delta_t(M_{12}) = \Delta_0(M_{12}) + \frac{\theta}{2}(P_1 \otimes P_3 - P_3 \otimes P_1),$$
$$\Delta_t(M_{13}) = \Delta_0(M_{13}) - \frac{\theta}{2}(P_1 \otimes P_2 - P_2 \otimes P_1)\,. \tag{4.2}$$

One can see from (4.1) that the generators of the stability group of $\theta_{\mu\nu}$, i.e. M_{01} which generates $O(1,1)$ and M_{23} which generates $\mathrm{SO}(2)$, both act through the primitive coproduct. Just as the preservation of translational symmetry is apparent from the primitive coproduct of the momentum generators (1.8), the invariance under the Lorentz subgroup $O(1,1) \times \mathrm{SO}(2)$ is indicated in the twisted Poincaré language by the unchanged coproducts of the corresponding generators. According to (4.2), the generators whose coproducts are deformed are those which mix the commutative directions with the noncommutative ones.

If we wish to discuss various invariances in the context of twisted Poincaré symmetry, we have to ensure that they hold under finite transformations, not only infinitesimal ones. To extend the concept of finite Poincaré transformations to the twisted case, one has to adopt the dual language of Hopf algebras: the algebra of functions $F(G)$ on the Poincaré group G, as a commutative algebra, is dual to $\mathcal{U}(\mathcal{P})$. The algebra $F(G)$ is generated by the elements $\mathbf{\Lambda}^\mu_\nu$ and $\mathbf{a}^\mu$, which are complex-valued functions, such that when applied to suitable elements of the Poincaré group, they would return the familiar real-valued entries of the matrix of finite Lorentz transformations, Λ^μ_ν, or the real-valued parameters of finite translations, a^μ. For example, if we consider the action of elements of $F(G)$ on a Lorentz group element $e^{i\omega^{\alpha\beta}M_{\alpha\beta}} \in G$ (without summation over α and β), we obtain

$$\mathbf{\Lambda}^\mu_\nu \left(e^{i\omega^{\alpha\beta}M_{\alpha\beta}} \right) = (\Lambda_{\alpha\beta}(\omega))^\mu_\nu\,, \tag{4.3}$$

$$\mathbf{a}^\mu \left(e^{i\omega^{\alpha\beta}M_{\alpha\beta}} \right) = 0, \tag{4.4}$$

while the action on translation group elements $e^{ia^\alpha P_\alpha}$ gives

$$\mathbf{\Lambda}^\mu_\nu \left(e^{ia^\alpha P_\alpha} \right) = 0 \,, \tag{4.5}$$

$$\mathbf{a}^\mu \left(e^{ia^\alpha P_\alpha} \right) = a^\mu \,. \tag{4.6}$$

The duality is preserved after twisting, but with deformed multiplication in the dual algebra [4]. The deformed coproduct of the twisted Poincaré algebra $\mathcal{U}_t(\mathcal{P})$ turns into noncommutativity of translation parameters in the dual $F_\theta(G)$ [5, 54, 55]:

$$[\mathbf{a}^\mu, \mathbf{a}^\nu] = i\theta^{\mu\nu} - i\mathbf{\Lambda}^\mu_\alpha \mathbf{\Lambda}^\nu_\beta \theta^{\alpha\beta} \,,$$
$$[\mathbf{\Lambda}^\mu_\nu, \mathbf{a}^\alpha] = [\mathbf{\Lambda}^\mu_\alpha, \mathbf{\Lambda}^\nu_\beta] = 0, \quad \mathbf{\Lambda}^\mu_\alpha, \mathbf{a}^\mu \in F_\theta(G) \,. \tag{4.7}$$

The "coordinates" x^μ, generating the algebra of functions with $\star$-product $\mathcal{C}_\theta$, transform by the coaction of the quantum matrix group (see, e.g., ref. [11], p. 61):

$$\delta : \mathcal{C}_\theta \to F_\theta(G) \otimes \mathcal{C}_\theta \tag{4.8}$$

as

$$(x')^\mu = \delta(x^\mu) = \mathbf{\Lambda}^\mu_\alpha \otimes x^\alpha + \mathbf{a}^\mu \otimes 1 \,. \tag{4.9}$$

The role of the deformed multiplication of "translation parameters" is to preserve the commutation relation of "coordinates" of the quantum space,

$$[x'_\mu, x'_\nu] = i\theta_{\mu\nu} \,, \tag{4.10}$$

the products being of course taken with the appropriate multiplication in $F_\theta(G) \otimes \mathcal{C}_\theta$.

Due to the nontrivial commutation relations (4.7), in the twisted case the functions $\mathbf{a}^\mu$ are no more complex-valued (though $\mathbf{\Lambda}^\mu_\nu$ still are, and satisfy (4.3) and (4.5)). However, there are elements of the Poincaré group for which the values of the functions $\mathbf{a}^\mu$ are still commutative. Such simple cases are the translations and the trivial Lorentz transformations, $\Lambda^\mu_\nu = \delta^\mu_\nu$, but they are not the only ones. For definiteness, let us consider various relevant finite twisted Lorentz transformations, as follows (again, the θ-matrix is assumed as in (3.1)):

(i) A boost in the commutative direction x^1:

$$\Lambda_{01} = \mathbf{\Lambda} \left(e^{\omega^{01}(\alpha)M_{01}} \right) = \begin{pmatrix} \cosh\alpha & \sinh\alpha & 0 & 0 \\ \sinh\alpha & \cosh\alpha & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \,; \tag{4.11}$$

(ii) A rotation between the noncommutative coordinates, x^2 and x^3:

$$\Lambda_{23} = \mathbf{\Lambda} \left(e^{\omega^{23}(\gamma)M_{23}} \right) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \cos\gamma & \sin\gamma \\ 0 & 0 & -\sin\gamma & \cos\gamma \end{pmatrix} \,; \tag{4.12}$$

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[4]A basic property of the duality is that the coproduct and multiplication of the deformed Hopf algebra directly influence the multiplication and coproduct, respectively, of the deformed dual Hopf algebra (see, e.g., refs. [9 – 11]).

(iii) A rotation between a commutative and a noncommutative coordinate, x^1 and x^2:

$$\Lambda_{12} = \mathbf{\Lambda}\left(e^{\omega^{12}(\beta)M_{12}}\right) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\beta & \sin\beta & 0 \\ 0 & -\sin\beta & \cos\beta & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \tag{4.13}$$

Using the general commutation relations (4.7) applied to the corresponding elements of the Lorentz group, we obtain in the cases *(i)* and *(ii)*

$$[\mathbf{a}^\mu(g), \mathbf{a}^\nu(g)] = [a^\mu, a^\nu] = 0\,, \tag{4.14}$$

where g is either $e^{\omega^{01}(\alpha)M_{01}}$ or $e^{\omega^{23}(\gamma)M_{23}}$, respectively. (This result holds also when time is noncommutative and the θ-matrix contains a nontrivial block in the upper left corner, $\theta_{01} = -\theta_{10} = \theta'$.)

In the case *(iii)*, when commutative and noncommutative coordinates mix, we obtain

$$\begin{aligned} \left[\mathbf{a}^2\left(e^{\omega^{12}(\beta)M_{12}}\right), \mathbf{a}^3\left(e^{\omega^{12}(\beta)M_{12}}\right)\right] &= i\theta(1 - \cos\beta)\,, \\ \left[\mathbf{a}^1\left(e^{\omega^{12}(\beta)M_{12}}\right), \mathbf{a}^3\left(e^{\omega^{12}(\beta)M_{12}}\right)\right] &= -i\theta\sin\beta\,, \end{aligned} \tag{4.15}$$

all the other commutators being zero. One can check that for all other Lorentz transformations mixing commutative and noncommutative directions, nontrivial commutators of "translation parameters" arise.

Once more it appears that in the twisted Poincaré context, the Lorentz transformations corresponding to the stability group of $\theta_{\mu\nu}$ behave just as in the commutative case, while the Lorentz transformations mixing the commutative and noncommutative directions require peculiar noncommuting translations. Remark that we imposed the Lorentz transformation *(iii)*, and we ended up with accompanying noncommuting translations showing up as the *internal mechanism* by which the twisted Poincaré symmetry keeps the commutator (4.10) invariant.[5] While one can still conceive an abstract geometrical meaning for the transformed generators $x'^\mu \in F_\theta(G) \otimes \mathcal{C}_\theta$, it is a conceptual challenge to confer them a physical meaning.

4.2 Fields in noncommutative space-time

Equipped with these results, let us return to the concept of a noncommutative field and the action of twisted Poincaré transformations on it. It was proposed in [53] that the construction of noncommutative fields should be started (as in commutative theories) from

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[5]We can view these translations as taking upon themselves the noncommutativity which would be naturally bestowed on the combination of commutative and noncommutative coordinates. For example, in our case, performing the Lorentz transformation (4.13) in the x^1, x^2-plane would at first sight seem to make both coordinates $x'^1 = x^1\cos\beta + x^2\sin\beta$ and $x'^2 = -x^1\sin\beta + x^2\cos\beta$ noncommutative. With the already noncommuting $x'^3 = x^3$, this would have given three noncommuting directions in the new system of reference, and two nontrivial commutators, $[x'^2, x'^3]$ and $[x'^3, x'^1]$. However, the twisted Poincaré symmetry enforces the appearance of the noncommuting translations (4.15), which reduce the number of nontrivial commutators back to one, $[x'^2, x'^3] = i\theta$ (as in (4.10)).

first principles, i.e. the general theory of induced representations (see, e.g., ref. [56]). In the commutative case, a classical field is a section of a vector bundle induced by some representation of the Lorentz group. The natural generalization of this construction is not succesful in the noncommutative case, mainly because the universal enveloping algebra of the Lorentz Lie algebra is not a Hopf subalgebra of the twisted Poincaré algebra. As a result, Minkowski space $\mathbb{R}^{1,3}$, which in the commutative setting is realized as the quotient of the Poincaré group by the Lorentz group, G/L (in an obvious notation), has no noncommutative analogue. For all mathematical details and the subtle points of the comparison between the commutative and noncommutative cases, we refer the reader to ref. [53]. In the same paper a way out was proposed, which retains Minkowski space but uses finite dimensional $\mathcal{U}(\mathcal{P})$-modules with trivial action of all momentum generators P_μ instead of finite dimensional Lorentz-modules.

While entirely agreeing with the analysis of ref. [53], we would like to propose here still another interpretation of the noncommutative field, which is closer to the implications of the dynamical calculations. Eqs. (4.7), and in particular (4.14) and (4.15), show how destructive the Lorentz transformations mixing commutative and noncommutative directions are for the coordinates: the coordinates become objects belonging to $F_\theta(G) \otimes \mathcal{C}_\theta$, to which one can not assign any numbers. The Minkowski space in the noncommutative setting appears not to have the same deep meaning to which we are used in Special Relativity, because the commutative and noncommutative coordinates have distinct properties. Our proposal is, therefore, to give up the Minkowski space $\mathbb{R}^{1,3}$ in favour of $\mathbb{R}^{1,1} \times \mathbb{R}^2$, but to retain the finite dimensional Lorentz-modules in the constructions of noncommutative fields.

Specifically, a commutative relativistic field has to carry a representation of the Lorentz group and at the same time to be a function of the space-time coordinates $x^\mu \in \mathbb{R}^{1,3}$ [6]. The consistent construction, such that the actions of the Poincaré group can be defined on the field, is achieved by the method of induced representations. The commutative field turns out to be an element of $C^\infty(\mathbb{R}^{1,3}) \otimes V$, where $C^\infty(\mathbb{R}^{1,3})$ is the set of smooth functions on Minkowski space and V is a Lorentz-module (a space of representations, bearing the actions of the Lorentz group). Since the field is defined as a tensor product, the action of the Lorentz group on it has to go through the coproduct, which in the commutative case is the primitive coproduct (1.5) and this is readily achieved since both $C^\infty(\mathbb{R}^{1,3})$ and V admit actions of the Lorentz group.

In the case of twisted Poincaré algebra, when trying to act with a Lorentz generator on an element of $C^\infty(\mathbb{R}^{1,3}) \otimes V$,

$$\Phi = \sum_i f_i \otimes v_i , \quad f_i \in C^\infty(\mathbb{R}^{1,3}) , \quad v_i \in V , \tag{4.16}$$

one has to use the twisted coproduct and at this point the procedure fails. The twisted coproduct of Lorentz generators (4.2) contains terms which require the action of the *momentum* operator on the elements of V, but V - a Lorentz-module - does not admit the

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[6]This statement and the argumentation below it are presented in an intuitive manner, disregarding mathematical rigour. For a rigorous treatment we refer the reader to ref. [53].

action of P_μ. This is why it was proposed in [53] to replace the Lorentz-module by a $\mathcal{U}(\mathcal{P})$-module with trivial actions of the momentum generators. The consequences of this construction are found in [53].

We propose as a simpler solution to retain V as a Lorentz-module, but to simply discard the action of all those Lorentz generators which are not allowed because of the additional terms containing the inadmissible momentum generator P_μ. Recall from (4.1) and (4.2) that the generators of the stability group of $\theta_{\mu\nu}$ still act via the primitive coproduct, therefore their action on elements of V is not prevented in any way. Their algebra also closes (it is the Abelian algebra $o(1,1) \times o(2)$).

To conclude, we propose that the *noncommutative field* be in $C^\infty(\mathbb{R}^{1,1} \times \mathbb{R}^2) \otimes V$, thus *carrying representations of the full Lorentz group, but admitting only the action of the generators of the stability group of $\theta_{\mu\nu}$, i.e. $O(1,1) \times$ SO(2)*.[7]

The generalization of this statement to the quantum case is straightforward: the field $\hat{\Phi} = \sum_i \hat{f}_i \otimes v_i$ becomes an operator through $\hat{f}_i = \hat{A}_i \otimes g_i$ which belong to $\mathcal{O} \otimes C^\infty(\mathbb{R}^{1,1} \times \mathbb{R}^2)$, where $\mathcal{O}$ is an algebra of field operators acting on the Hilbert space of states. The product of the field operators is not influenced by the twist, while the functions of $C^\infty(\mathbb{R}^{1,1} \times \mathbb{R}^2)$ are multiplied by the $\star$-product:

$$(\hat{A} \otimes g)(\hat{B} \otimes h) = \hat{A}\hat{B} \otimes g \star h, \quad \hat{A}, \hat{B} \in \mathcal{O}, \quad g, h \in C^\infty(\mathbb{R}^{1,1} \times \mathbb{R}^2). \tag{4.17}$$

The Lorentz-module V is in no way affected by the quantization. What is different compared with the commutative case is that now the field picks up its x-dependence from $\mathbb{R}^{1,1} \times \mathbb{R}^2$ instead of the Minkowski space, which is in full agreement with the dynamical calculations. Again, only the action of the generators of the stability group of $\theta_{\mu\nu}$ is allowed and it goes through the primitive coproduct. Since the quantum field $\hat{\Phi}$ carries a representation of the Lorentz group through v_i, the field operators $\hat{A}_i$ will carry in their turn corresponding Lorentz representation indices. This, together with the usual product in the algebra of operators $\mathcal{O}$, make the Hilbert space of states (in essence, the Fock space) identical to the one of the commutative QFT[8].

5. Conclusions

In this paper we have studied the confrontation of the Lorentz symmetry, the residual $O(1,1) \times$ SO(2) symmetry and the twisted Poincaré symmetry in noncommutative QFT with constant antisymmetric parameter $\theta_{\mu\nu}$. Based on Pauli's Theorem [36, 37], we concluded that the Lorentz group can not provide a symmetry for NC QFT. We have presented

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[7]Loosely stated, the difference between the approach of ref. [53] and the present one is the following: while in ref. [53] the noncommutative fields were induced by a part of the representations of the Lorentz group, but carrying the action of all the generators of the Poincaré algebra through the twisted coproduct, in this paper we advance the idea of having the noncommutative fields induced by all the representations of the Lorentz group, but carrying only the action of the generators of the stability group of $\theta_{\mu\nu}$. An advantage of the latter approach is that the finite transformations of the noncommutative fields are readily obtained.

[8]Consequently, the spin-statistics relation holds just as in the commutative case.

a new dynamical result, the Tomonaga-Schwinger equation in the interaction picture of NC QFT, which supports the previous computations in various models, showing the infinite nonlocality in the noncommutative directions, the emergence of the light-wedge causality condition and the symmetry of NC QFT under the stability group of $\theta_{\mu\nu}$, $O(1,1) \times SO(2)$. This result is general and of significance for building up concrete noncommutative models.

Persuaded that the dynamical calculations and the symmetry arguments have to match each other in NC QFT as in any other physical theory, we embarked upon deepening our understanding of what is meant by twisted Poincaré invariance. Following the proposal of ref. [53] to approach the definition of the noncommutative fields starting from the method of induced representations, we proposed in section 4.2 a new interpretation for the noncommutative fields. With this construction, the meaning of the twisted Poincaré symmetry in NC QFT becomes transparent: it represents actually the invariance with respect to the stability group of $\theta_{\mu\nu}$, while the quantum fields still carry representations of the full Lorentz group and the Hilbert space of states has the richness of particle representations of the commutative QFT.

Thus, the twisted Poincaré symmetry and the invariance under the stability group of $\theta_{\mu\nu}$ peacefully coexist in NC QFT. Lorentz symmetry can not be achieved with constant noncommutativity parameter, therefore noncommutative QFT can not be interpreted as indistinguishable from commutative QFT.

Acknowledgments

A. Light-wedge configuration

Instead of showing that, in general,

$$[\mathcal{H}_{\text{int}}(x)_\star, \mathcal{H}_{\text{int}}(y)_\star] \tag{A.1}$$

does not vanish identically for $(x^0 - y^0)^2 < (\mathbf{x} - \mathbf{y})^2$ we choose a simpler commutator

$$\big[\phi(x), \phi(y) \star \phi(y)\big].$$

We know that it vanishes for $(x^0 - y^0)^2 < (x^1 - y^1)^2$, but we now want to show that it does not necessarily vanish for $(x^0 - y^0)^2 < (\mathbf{x} - \mathbf{y})^2$.

For this purpose we consider it in the form

$$\big[\phi(x), \phi(y) \star \phi(y)\big] = i\Delta(x - y) \star \phi(y) + i\phi(y) \star \Delta(x - y). \tag{A.2}$$

Writing the field in terms of the Fourier transform

$$\phi(y) = \sum_l e^{il\cdot y}\phi(l)\,, \tag{A.3}$$

$$\sum_l e^{il\cdot y}\big[\Delta(x^0 - y^0, x^1 - y^1, x^2 - y^2 + \theta l_3, x^3 - y^3 - \theta l_2)+ \tag{A.4}$$

$$+ \Delta(x^0 - y^0, x^1 - y^1, x^2 - y^2 - \theta l_3, x^3 - y^3 + \theta l_2)\big]\,.$$

For this to vanish the quantity in brackets should vanish for every l since all $\phi(l)$ are linearly independent. We will now show that this does not happen in a special configuration, where $x^1 - y^1 = x^3 - y^3 = 0$ and $l_2 = 0$, which implies the light cone condition $(x^0 - y^0)^2 < (x^2 - y^2)^2$.

Let us choose $x^2 - y^2 + \theta l_3 = 0$. Now in the first term of (A.3) all the space coordinates vanish and $-(x^0 - y^0)^2 < 0$, i.e. the vector is timelike and this term survives in the integration. Therefore the commutator $\big[\phi(x), \phi(y) \star \phi(y)\big]$ does not vanish for $(x^0 - y^0)^2 < (\mathbf{x} - \mathbf{y})^2$ even though it vanishes for

$$(x^0 - y^0)^2 < (x^1 - y^1)^2$$

and we can conclude that the commutator does not vanish for a space-like separation if the light-wedge condition is not met.

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